HEGEL, MARX AND THE CALCULUS

by C. Smith

1. Marx's Mathematical Work

In the preface to the second edition of *Anti-Dühring*, Engels referred to the mathematical manuscripts that Marx had left, and said that they were extremely important. But they remained inaccessible for fifty years, only being published in Russian translation in 1933. In 1968, they were first made available in their original form, in the Russian edition from which the present volume has been translated. To this day, very little attention has been paid to them.*

But despite this, Engels's assessment was right. Marx spent a great part of the last few years of his life on this work which must be seen, not as a curiosity of mathematical history, but as a significant contribution to the development of dialectical materialism.

Marx was not a mathematician. In the course of his work on *Capital*, he continually strove to overcome his lack of knowledge in this field, so that he could apply algebraic methods to quantitative aspects of political economy. But, from 1863, his interest turned increasingly to the study of infinitesimal calculus, not just as a mathematical technique, but in relation to its philosophical basis. By 1881, he had prepared some material on this question, and this forms the greater part of this volume. It is clear that these manuscripts were not intended for publication, being aimed at the clarification of Engels and himself. Not only is the first manuscript marked 'For the General' and the second 'Für Fred', but they are written in that mixture of German, English and French in which the two men usually communicated.

Much ink has been spilled in recent years to try to show that Marx did not agree with Engels's work on the natural sciences. These efforts

* See D.J. Struik, 'Marx and Mathematics', *Science and Society*, 1948, pp.181-196. V. Glivenko, *Der Differentialbegriff bei Marx und Hadamard*', *Unter dem Banner des Marxismus*, 1935, pp.102-110.

are part of the hostility to the idea of the dialectics of nature and the general attack on dialectical materialism as a whole. They never had any basis in the published writings of Marx, or in his correspondence with Engels. These manuscripts show, apart from anything else, that Engels's work was part of a joint project on the part of the two founders of materialist dialectics.

When we read the letter in which Engels gave his reaction to them, we get a clue to their real significance.* Engels comments: 'Old Hegel guessed quite correctly when he said that differentiation had for its basic condition that the variables must be raised to different powers, and at least one of them to at least the second . . . power.' Leaving aside for the moment the mathematical meaning of this remark, it directs our attention to the connection of Marx's work with its point of origin: Hegel's *Science of Logic*, especially the section on *Quantitative Infinity* (Miller translation, pp.238-313). Engels knows that this is what Marx is referring to, without Hegel's name being mentioned.

It is surprising that the editors of the manuscripts, who have been so painstaking in following up all Marx's mathematical references, should have ignored this quite unmistakable connection. While the conclusions of Hegel and Marx reflect the conflict between idealism and materialism, of course, they discuss the same issues and refer to many of the same authors.† It is worth noting that, while Hegel often stresses his opinion that mathematical forms are quite inadequate for the expression of philosophical ideas, he nonetheless spends about one-eighth of the *Science of Logic* on the question of mathematics, most of this in relation to calculus. Marx, on the other hand, never echoes Hegel's deprecatory attitude to mathematics.

2. *The Crisis of Infinity*

In the course of 2,500 years, mathematics has undergone a number of profound crises, all of which may be traced to the question of the infinite. Greek mathematics ran into this trouble in the 5th century BC, from two directions. The first was when Zeno produced his famous paradoxes.§ Apparently his aim was to justify the contention

* Engels to Marx, August 10, 1881. See page xxvii-xxx for a translation of this letter and two other items from the Marx-Engels correspondence.

† Perhaps Marx's references to Newton's *Principia* were prompted by those of Hegel. His references to John Landen certainly were.

§ See Lenin, *Collected Works*, Vol.38, pp.256-260.

of his master Parmenides, that Being is one and unchanging, by showing that multiplicity and motion led to contradiction, and were therefore mere appearance.

All four of Zeno's paradoxes — 'Achilles and the Tortoise', 'The Arrow', 'The Dichotomy' and 'The Stadium' — turn on the problems of the infinitely small magnitude and the infinitely large number. They demonstrate that movement is contradiction, as is the indefinite divisibility of space and time.

Soon after they were launched on the academic world, it was shaken by a second bombshell. The followers of Pythagoras believed that number — and that meant the set of integers 1, 2, 3 . . . — was the fundamental basis of all Being. But the geometrical theorem named after their leader showed that the lengths of certain lines, for example the diagonal of a square exactly one unit in size, could not be measured in terms of integers. Today we would say that $\sqrt{2}$ is not a rational number. They tried to keep this scandal a secret, but the terrible news got out.

It is easy to see that this trouble also springs from the infinite, if you try to write down as a decimal the number whose square is exactly 2. Greek mathematics evaded the question of infinity from then on, by restricting its attention to the relations between lines, areas and volumes, without ever attempting to reach a general conception of *number*.

It was partly in response to these problems of infinite divisibility that the Ionian philosophers — Europe's first physicists — developed their conception of the atoms, indivisible pieces of matter constantly moving in the void. This concept, revived after 2,000 years, became the foundation for the mechanistic science of Galileo and Newton. As we shall see, this attempt to avoid the contradictions of the infinitely divisible continuum could achieve its great successes only within definite limits.

Mathematics from the time of the Renaissance increasingly found itself facing the question of movement, and this confrontation led in the seventeenth century to the emergence of the algebraic geometry of Descartes and of the calculus.* Movement meant that the moving object had to pass through 'every point' of a continuous interval.

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* Boyer, *The History of Calculus*, is still the best account. Baron, *The Origins of the Calculus*, is more detailed on the period before Newton and Leibnitz. For a useful brief account, see Struik, *A Concise History of Mathematics*.

Science would not escape the problem of sub-dividing the interval 'indefinitely' into 'infinitely small' pieces. Up to the time Hegel was writing (1813), mathematicians freely operated with such objects, adding them up as if they were ordinary numbers. Sometimes they obtained results which were correct and useful, and sometimes they obtained nonsense in algebra.

Newton had to express in mathematical form the concept of instantaneous velocity. If an object is moving with uniform speed, this is easy: simply divide the distance travelled by the time it took to cover it. But what can be said about an object which is speeding up or slowing down? We must find the *average* speed over some time interval, and then consider smaller and smaller intervals. But to obtain the velocity 'at an instant' would entail dividing 'an infinitely small distance' by an 'infinitely small' time. It would be the 'ratio of vanishing magnitudes'.

Earlier writers, notably Galileo's pupil Cavalieri, had written of 'indivisibles', objects without length, which, when taken in infinite number, somehow made up a finite length. Newton refused to take this way out. The numerator and denominator of this ratio had to be 'vanishing divisibles'. The distance travelled, say x, he called a 'fluent', while its rate of change or instantaneous velocity he called its 'fluxion', denoted $\dot{x}$. A 'moment' of time t he denoted 'o' — not to be confused with 0 — so that the distance travelled during this moment was $\dot{x}o$. The $\dot{x}$ was the 'ultimate ratio' between them which, he said, had to be understood 'not as the ratio before they vanish or afterwards, but with which they vanish'. Only then could their powers — squares, cubes, etc. — be taken as zero, or 'neglected'. Both Newton and Leibnitz who originated the differential calculus independently at the same period, struggled to explain what this meant. Leibnitz invented the now standard notation 'dx', 'dt' for his 'differentials', whose ratio was the 'differential quotient' $\frac{dx}{dt}$ .No wonder that Bishop Berkeley made the most of this obscurity — Marx was to call it 'mysticism' — to ridicule the Newtonians. He called their 'vanishing quantities' 'the ghosts of departed quantities' and asked how anyone who accepted such things could object to the mysteries of religion.*

* The full title of Berkeley's 1734 polemic, directed against Newton's follower Halley, is *The Analyst or a Discourse Addressed to an Infidel Mathematician. Wherein it is examined whether the object principles and inferences of modern analysis are more distinctly*

Of course, as an Englishman, Newton could get round the problem: 'everyone knew' that things moved and possessed a velocity at each instant of time. The contradictions of motion could be ignored. This has been described as 'empirical dogmatism', in contrast with the 'metaphysical dogmatism' of Leibnitz.

Throughout the eighteenth century the difficulty remained. Mathematics developed in leaps and bounds, but the careful and rigorous argumentation of the Greeks was thrown to the winds. The phrase of d'Alembert summed up the attitude of the time: *allez en avant et la foi vous viendra* (go ahead and faith will come). As great a mathematician as Euler can find himself trying to base the calculus on the multiplication and division of zeroes of different orders.*

3. *Hegel and the Infinite*

This is still the situation when Hegel takes up the issue. He condemns Leibnitz in particular for founding the calculus in a manner which was as 'non-mathematical as it is non-philosophical' (*op.cit.*, p.793).† His aim in discussing the subject is, he says, 'to demonstrate that the infinitely small . . . does not have merely the negative, empty meaning of a non-finite, non-given magnitude . . . but on the contrary has the specific meaning of the qualitative nature of what is quantitative, of a moment of a ratio as such'. (*op.cit.*, p.267) To see the significance of this, we must examine the part played by the ideas of 'finite' and 'infinite' in Hegel's work, as against the meaning given to them by Kant in particular.

For Kant, as for all bourgeois philosophy before Hegel, thought is the activity of individual human beings, limited in their knowledge and power of understanding by their own personal experience. These 'finite beings' cannot know things as they are 'in themselves', or the interconnections between separate things. We come into contact with unlimitedness, freedom, infinity, only when we obey the moral law, and even this refers only to *intention*, not to the actual consequences of

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conceived or more evidently deduced than religious mysteries and points of faith. 'First Cast the Beam Out of Thine Own Eye; and Then Shalt Thou See Clearly to Cast the Mote Out of Thy Brother's Eye'.

* E.T. Bell, in *The Development of Mathematics*, p.284, refers to 'The Golden Age of Nothing'. See Appendix III for a discussion of Euler's work.

† See also Lenin, *op.cit.*,, p.209.

the actions of finite beings. The infinite is and must always remain unattainable, never *actualised*.

Hegel spent his entire life fighting against this conception and exposing its implications, and this with a passion with which he is rarely credited. For him, the finite things we find in the world are united with the infinite, and the limited consciousness of individual people are elements of infinite Mind or Spirit. He condemned those subjective ways of thought which saw the world as just a collection of finite things, cut off from each other and from their totality.

Such an outlook could only look upon the infinite as the 'non-finite', beyond our reach. This 'bad' or 'spurious' infinite was 'what ought to be and is not', just the wearisome repetition of one finite thing after another, followed by an empty 'and so on'. Instead of all-sided necessity, subjectivism only sees the endless chain of cause and effect, and in place of the unlimited development of the human Spirit it knows only the separate experiences of isolated human atoms (*op.cit.*, pp.109-156).*

Spinoza had denied the scholastic '*infinitum actu non datur*' — 'there is no actual infinity'. He saw that to determine something, to set a boundary around it, was to negate everything else, and so to point beyond the boundary. Hegel applauded this but went a huge step further. The unity of the finite and the infinite was not something fixed, 'inert', but contained 'the negative unity of the self, i.e. subjectivity'. What Hegel calls 'Being-for-self' is the negation of the infinite back into the finite, thus the negation of negation, making the finite a part of the 'mutual determinant connection of the whole'. Hegel saw this as the basis of idealism, 'the fundamental notion of philosophy'. The isolated finite thing 'has no veritable being'; the negative element which lies at its heart is 'the source of all movement and self-movement'.†

Hegel develops this conception of the finite and the infinite in the course of his examination of Quality, 'the character or mode' of Being. He tries to show how 'Being-for-self suppresses itself. The qualitative character, which is the One or unit has reached the extreme point of its characterisation, has thus passed over into determinateness (quality) suppressed, i.e. into Being as Quantity.' In analysing Quantity, mag-

* Also *Phenomenology of Spirit*, Miller translation, pp.143-145; *Encyclopaedia*, Sections 93-95.

† *Encyclopaedia*, end of Sections 95. Also Lenin, *op.cit.*, pp.108-119

nitude (determinate quantity) and quantum (how much), he is concerned with 'an indifferent or external character or mode, of such a kind that a thing remains what it is, though its quantity is altered, and the thing becomes greater or less'. (*Encyclopaedia*, sections 104-105)

Common sense, of course, is happy to take the idea of number for granted. Hegel shows that it contains contradiction within it. 'Everybody knows' that quantum can be altered. But, says Hegel, 'not only *can* it transcend every quantitative determinateness, not only *can* it be altered, but it is posited that it *must* alter . . . Thus quantum impels itself beyond itself . . . The limit which again arises in this beyond is, therefore, one which simply sublates itself again and beyond to a further limit, *and so on to infinity*'. (*Science of Logic*, p.225)

In the 'bad infinity' of the alternation of a particular quality and its negation, we at least have the interest of the difference between its two terms. But in the endless sequence of quanta, each term is identical with its successor, determinateness having been suppressed. This Quantitative Infinite Progression moves towards infinity, but never gets any closer to it, says Hegel, 'for the difference between quantum and its infinity is essentially *not* a quantitative difference'. It is in this connection that Hegel discusses the calculus.

Hegel is deeply dissatisfied with the vagueness of the mathematicians about differentiation. Are the differentials dy, dx finite quantities, which can be divided into each other? Or are they zero? In that case their ratio would have no meaning — or any meaning you like to give it. But dy or dx are not 'quanta': 'apart from their relation they are pure nullities'. The mathematicians had tried to treat them as in 'an intermediate state . . . between being and nothing', but this cannot exist. For 'the unity of being and nothing . . . is not a *state* . . . on the contrary, this mean and unity, the vanishing or equally the becoming is alone their *truth*'. (*Science of Logic*, pp.253-254)

4. Marx and Engels on the Infinite

So Hegel's detailed examination of the calculus is not at all a digression, but an investigation of the way science and philosophy had dealt with questions which lay at the very basis of his outlook. Marx and Engels, as materialists, did not accept Hegel's idealism, of course. But in their negation of Hegel's system, they based themselves on this same view of the relation between the finite and the infinite, with its profoundly revolutionary implications. Where Hegel saw 'Spirit' as the 'infinite Idea', Marx grasped the infinite experience of humanity as the highest form of the infinite movement of matter. The development of human powers of production meant the continual penetration of this movement in all its continually-changing forms and interconnections.

The knowledge of each individual man or woman is limited, as is the knowledge of the entire race at any particular time. But in the struggle against nature, each finite person expresses in himself the unlimited potential of mankind to master nature, and through this the all-sided movement of matter of which he is a part.

That is why the positivist and the empiricist, who know only their own 'experience', face the for them insoluble 'problem of induction'. Since they can never live long enough to 'experience' the infinite — count it, or measure it, or classify it — they must deny its actuality. Consequently, they can never grasp the essential universality of a law, and are walled off from universal movement and all-sided interconnection.

Engels put the matter very clearly. He accepts the statement of the botanist Nägeli that 'we can know only the finite',

> 'in so far as only finite objects enter the sphere of our knowledge. But the proposition needs to be supplemented by this: "fundamentally we can know *only the infinite*". In fact all real, exhaustive knowledge consists solely in raising the individual thing in thought from individuality into particularity and from this into universality, in seeking and establishing the infinite in the finite, the eternal in the transitory. The form of universality, however, is the form of self-completeness, hence of infinity; it is the comprehension of the many finites in the infinite . . .
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> ' All true knowledge of nature is knowledge of the eternal, the infinite, and hence essentially absolute. But this absolute knowledge has an important drawback. Just as the infinity of knowable matter is composed of the purely finite things, so the infinity of thought which knows the absolute is composed of an infinite number of finite human minds, working side by side and successively at this infinite knowledge, committing practical and theoretical blunders, setting out from erroneous, one-sided and false premises, pursuing false, tortuous and uncertain paths, and often not even finding what is right when they run their noses

against it (Priestley). The cognition of the infinite is therefore beset with double difficulty, and from its very nature can only take place in an infinite asymptotic progress.' (*Dialectics of Nature*, pp.237-238)

'It is just *because* infinity is a contradiction that it is an infinite process, unrolling endlessly in time and in space. The removal of this contradiction would be the end of infinity. Hegel saw this quite correctly, and for that reason treated with well-merited contempt the gentlemen who subtilised over this contradiction.' (*Anti-Dühring*, pp.75-76)

5. *Marx and the Calculus*

In his mathematical work, Marx echoes Hegel's scorn for the vain efforts of the mathematicians to evade the contradictions inherent in motion, continuity and the infinity. But their attitudes to mathematics were quite opposed. For the objective idealist Hegel, mathematics, like natural science, occupied very lowly stages in the unfolding of the Idea. Mathematics, he thought, ought to be 'stripped of its fine feathers'. 'The principle of *magnitude*, of difference not determined by the Notion, and the principle of *equality*, of abstract lifeless unity, cannot cope with that sheer unrest of life and its absolute distinction . . . Mathematical cognition . . . as an external activity, reduces what is self-moving to mere material, so as to possess in it an indifferent, external, lifeless content.'*

But Marx sees that mathematical abstractions, purely formal as they must necessarily appear, contain knowledge of self-moving matter, knowledge of generalised relationships between material objects which is ultimately abstracted from social practice, and which is indispensable for practice.

Hegel and Marx are each concerned to express the contradiction of movement and change, as Hegel says, to 'really solve the contradiction revealed by the method instead of excusing it or covering it up'. (*Science of Logic*, p.277)

Where Hegel only needs to expose the false methods of thought which underly these ambiguities, Marx feels impelled to go deeper into the mathematical techniques themselves and provide an alter-

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* *Phenomenology*, p.27 See pp.24-26. Also *Encyclopaedia* Sections 259, 267 (*Philosophy of Nature*).

native. He wants to be able to develop the derivative $\frac{dy}{dx}$, not as an approximation, but as an expression of the actual *motion* of the function $f(x)$.

Unlike Hegel, Marx refers to the work of d'Alembert on this question (see Appendix IV, p.165). He had not resolved the problem, but had drawn attention to the weakness of existing mathematical methods: its lack of a clear conception of *limit*. Marx attempts to answer this by the following means, which we summarise in modern notation.

If we want to differentiate a function $f(x)$, proceed as follows: take x_1 different from x and subtract the expression for $f(x)$ from that for $f(x_1)$. Let us call this $F(x, x_1) = f(x_1) - f(x)$, a function of *two* variables x and x_1. Now express $F(x,x_1)$, if possible, as $(x_1 - x)G(x,x_1)$. Finally, in the function G, set $x_1 = x$, and call $G(x,x) = f'(x)$, the derivative function. In this way, we avoid all trouble with 'infinitely small quantities'. Those puzzling differentials now have meaning only in the relationship $df(x) = f'(x)dx$. (Marx assumes without good reason that G will always be continuous at $x_1 = x$).

Illustrating this with a simple example, take $f(x) = x^3$,

$$x_1^3 - x^3 = (x_1 - x)\ (x_1^2 + x_1x + x^2)\ ,$$

$$\text{so}\ \ G(x,x_1) = x_1^2 + x_1x + x^2\ ,$$

$$\text{leading to}\ f'(x) = G(x,x) = 3x^2.$$

We should miss the whole point of this, however, if we did not heed Marx's remark at the start of the first manuscript: 'First making the differentiation and then removing it therefore leads literally to *nothing*. The whole difficulty in understanding the differential operation (as in the *negation of the negation* generally) lies in seeing *how* it differs from such a simple procedure and therefore leads to real results.' Marx is referring to the operations of first making x_1 different from x, and then making it the same as x once more. For only through this double negation is the actual *movement* of $f(x)$ registered in the derivative $f'(x)$. This is the idea expressed by Hegel (and referred to by Engels in his letter to Marx quoted above) when Hegel says that 'the calculus is concerned not with variable magnitudes as such but with the *relations of powers* . . . the quantum is genuinely completed into a qualitative reality; it is posited as actually infinite.' (*Science of Logic*, p.253)

Hegel's comments on calculus were made just at the point when mathematics was about to make a fresh effort to tackle these issues. (*The Science of Logic* was published in 1813). During the next 70 years, the basic concepts of function, limit and number were completely transformed. But these new ideas were not known to Marx. As this volume makes clear, his knowledge was drawn from textbooks which, although they were still in use in his time, did not reflect the newer developments.*

But this does not mean that the work of Marx and Hegel was rendered valueless as a result of these changes, for the further expansion of mathematical knowledge to this day continually encounters the same problems, but at a deeper level.

6. *Later Developments*

When mathematicians before 1830 spoke of a *function*, what they had in mind was roughly what Euler had described in the words: 'some curve described by freely leading the hand'. Lagrange took it for granted that such a 'smooth' object would have a 'Taylor expansion': $a + bx + cx^2 + dx^3 \ldots$, and called it 'analytic'. (The method advocated by Marx will only work for such functions.) The more general modern conception of functional relationship was clarified by Dirichlet and others in the 1830s. It simply meant that to each of a given set of values of x corresponded a given value $f(x)$.

It was in 1821 and 1823 that Cauchy published his books which attempted to give a logical definition of *limit*. These ideas were tightened up by Weierstrass in the 1860s. Now, to say that a function $f(x)$ tended to a limit as x tended to x_0, meant the following: there exists a number L such that, for any positive quantity ε, however small, there exists a quantity δ, such that whenever

$$x_0 - \delta < x < x_0 + \delta,\ L - \varepsilon < f(x) < L + \varepsilon\ .$$

Using this idea, it was possible to define continuity, and understand the derivative $f'(x)$ as the limit of $\frac{f(x+\delta) - f(x)}{\delta}$, as δ tended to 0.†

* To this day, students are introduced to calculus with the aid of arguments drawn essentially from the 18th century. The book by Lacroix, which Marx made so much use of, was still being reissued in 1881.

† These ideas, as well as those of Cantor, were to some extend anticipated in 1820-40 by the Bohemian priest Bolzano, although his work was not generally appreciated until later.

Could mathematicians now say that they had returned to the rigour of argumentation of their Greek predecessors, but at the same time grasped the nettle of infinity? Was the new form of analysis able to dispense with intuitive ideas of space and time? Not yet.

For the idea of 'limit' was still infected with intuition in the shape of the continuous collection of numbers contained in the interval between the two values. Weierstrass's definitions aimed to provide a *static* framework for what was essentially dynamic. Together with Dedekind and others, he grappled with the *continuum* of numbers, clarifying many of the concepts of modern analysis. Then, in 1872, Cantor's work appeared, which tried for the first time to deal rigorously with infinite sets of objects, to count the actually infinite, and to provide a consistent arithmetic of 'transfinite numbers'.*

In 1900, the leading figure in world mathematics, Henri Poincaré, could confidently declare that 'absolute rigour has been attained'. As Bell reports him, Poincaré was quite certain that 'all obscurity had at last been dispelled from the continuum of analysis by the nineteenth century philosophies of number based on the theory of infinite classes . . . All mathematics, he declared, had finally been referred to the natural numbers and the syllogisms of traditional logic; the Pythagorean dream had been realised. Henceforth, reassured by Poincaré, timid mathematicians might proceed boldly, confident that the foundation under their feet was absolutely sound.' (Bell, *The Development of Mathematics*, p.172. See also p.295.)

How wrong he was! In the early years of this century, the geometry of Euclid, thought by Kant and nearly everyone else to be founded on self-evident truths, was shown to be not the correct description of actual space; even worse, the foundations of logic itself began to shake. These problems of the foundation of mathematics and logic were directly linked to the paradoxes of infinite sets.

Throughout this century, the search for an uncontroversial basis for mathematical science has produced the sharpest controversy. In the attempt to evade the paradoxes of the infinite, two opposite trends have been at war. On the one side stand the *formalists*, constantly trying to see mathematics as a game played with undefined symbols, having no more meaning than chess. By setting out the rules of this game in the form of consistent axioms, all the relations between the

* But while Cantor believed the infinitely large was actual he absolutely denied the existence of the actually infinitely small.

invented objects of the game can be worked out. Then, in 1931, disaster struck, in the form of the theorem of Gödel: he showed that the game called arithmetic could produce well-formulated problems which were *undecidable* within the system.

Against the formalists stood the *intuitionists*, led by Brouwer and Heyting, tracing their origins back to Kant. For them, mathematics had at its basis certain unanalysable concepts which were given *a priori*. Infinity was not among them, and mathematics had to be reconstructed after expunging reference to such monsters.

7. *What is mathematical knowledge?*

These controversies appear to be of interest only to those engaged in the mathematical game. In fact, however, the crisis which still wracks the foundations of physics turns precisely on the contradictions of the discrete and the continuous, the finite and the infinite. Some physicists have been led to consider the possibility of a 'finitistic mathematics' as a way out of their troubles.*

Marx's work on calculus did not only concern the problems of infinitesimals. Having explained his 'algebraic method' of differentiating, he takes a further step which brings him very close to the spirit of twentieth century mathematics. He describes the further development of calculus in terms of a reversal of roles, in which the symbols for the differential coefficient are transformed into 'operational formulae' (*Operationsformel*), satisfying 'operational equations'. These ideas give a basis for a materialist conception of mathematical knowledge which is of great importance for dialectical materialism as a whole. For mechanical materialism, formal abstractions carry great dangers. They are taken in isolation from the movement from living perception to social practice, and the entire process is seen in reverse, rather like the negative of a photograph. For the abstract symbol is mistaken for the actual object of knowledge, while the concrete object is seen only as mere background.

Modern mathematics has generalised the processes of algebra into stratospheric levels of abstraction, where the objects of the science seem to be completely undefined. All that we know about them is the rules which govern their relationships to each other, and these seem to be decided by the will of the mathematician. Empiricists are then

* See Weizsäcker, *The World View of Physics*, Chapter 5. Also his contributions to T. Bastin (ed) *Quantum Theory and Beyond*.

puzzled by the apparent coincidence which makes precisely these abstract forms express the relationships of material processes. Marx's approach to the calculus, however, shows the dialectical relationship between the abstract symbols and the movement of matter from which they have been abstracted.

In discussing the nature of abstraction, Hegel attacks those views which place the abstract on a lower level than 'sensuous, spatial and temporal, palpable reality'. 'In this view, to abstract means to select from the concrete object for *our subjective purposes this or that mark*'. (*Science of Logic*, p.587, Lenin *op.cit.*, pp.170-171).

Hegel — from his idealist standpoint, of course — thinks on the contrary that 'abstract thinking. . . is not to be regarded as a mere setting aside of the sensuous material, the reality of which is not thereby impaired; rather it is the sublating and reduction of that material as mere *phenomenal appearance* to the *essential*.' (*Science of Logic*, p.588) Hegel cannot allow these considerations to apply to mathematics, which he regards as being unable to capture the richness of movement and interconnection. Marxism, turning the dialectic on to its material feet, grasps the way that mathematical abstractions, seen in the context of the entire development of natural science and technology, can contain real knowledge of the movement of matter. This is the meaning of Engels's description of mathematics as 'an abstract science which is concerned with creations of thought, even though they are reflections of reality'. (*Dialectics of Nature*, p.218)

To the modern student of mathematics, these manuscripts of Marx have, no doubt, an archaic appearance. But we have seen that the questions with which they really deal are infinity, the relation between thinking and being, and movement, the central philosophical issues. As our brief look at the history of mathematics has shown, it is just these questions which underlie the crisis which still wracks the foundations of mathematics. These difficulties are linked with the methodological problems facing many other branches of science, problems which deepen with every major scientific advance.

A century ago, Marx and Engels paid particular attention to the development of natural science and mathematics, precisely because they knew that dialectical materialism could only live and grow if it based itself on the most up-to-date discoveries of science and concerned itself with the problems which these entailed for fixed, 'common sense' views of reality. Today, this is still more vital than when

Engels was preparing his articles against Dühring and his notes on the dialectics of nature, and when Marx was writing these mathematical manuscripts.

When we look at this work as a whole, another common feature is striking: the way Marx and Engels return to Hegel for clarification. Marxism is the negation of absolute idealism — but in the Hegelian sense of simultaneous abolition and preservation. Contrary to the contention of various revisionist schools, Marx did not make a single, once-for-all break with Hegel, but continuously returned to Hegel to negate his idealism, as did Lenin and Trotsky after him.

These manuscripts, therefore, may be seen as the last of Marx's returns to Hegel. They should be a spur to the Marxists of today to take forward the fight for the dialectical materialist method in connection with the latest developments in mathematics and natural science through a still deeper struggle with Hegel.